

MODULI OF REPRESENTATIONS, QUIVER GRASSMANNIANS, AND HILBERT SCHEMES

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ABSTRACT. It is a well established fact, that any projective algebraic variety is a moduli space of representations over some finite dimensional algebra. This algebra can be chosen in several ways. The counterpart in algebraic geometry is tautological: every variety is its own Hilbert scheme of sheaves of length one. This holds even scheme theoretic. We use Beilinson's equivalence to get similar results for finite dimensional algebras, including moduli spaces and quiver grassmannians. Moreover, we show that several already known results can be traced back to the Hilbert scheme construction and Beilinson's equivalence.

1. INTRODUCTION

Assume k is an algebraically closed field and X is a projective subscheme of \mathbb{P}^n defined by some homogeneous equations f_1, \dots, f_r in $k[X_0, \dots, X_n]$. We want to realize X as a moduli space of quiver representations and as a quiver grassmannian in a natural way. Moreover, we also like to have a construction making the quiver as small as possible. Let A be a bounded path algebra kQ/J , where Q is a finite quiver and J is an ideal of admissible relations in the path algebra kQ . Moduli spaces for quiver representations have been defined by King in [11], a quiver grassmannian is just the variety of all submodules of a given module M of a fixed dimension vector. We note that we can consider moduli spaces and also quiver grassmannians with its natural scheme structure. Moreover, any quiver grassmannian is a moduli space (just a moduli space of submodules of a given module), and there are natural morphisms between quiver grassmannians and moduli spaces. Under certain additional conditions these morphisms are even isomorphisms. Those isomorphisms are always hidden in our construction. Since these morphisms can be seen explicitly in our construction we do not need any general result for those morphisms. This is the main reason why we use line bundles in our construction, for arbitrary vector bundles all constructions become much more technical. The other advantage of using line bundles is that we can always use modules of dimension vector $(1, 1, \dots, 1, 1)$ (also called thin sincere).

Theorem 1.1. *Let X be any projective scheme defined by equations f_1, \dots, f_r in a projective n -space \mathbb{P}^n . Then there exists a quiver Q , an ideal J in the path algebra kQ and a kQ -module M so that*

- (1) *X is isomorphic to the moduli space of all indecomposable kQ/J -modules of dimension vector $(1, \dots, 1)$, and*
- (2) *X is isomorphic to the quiver grassmannian of all submodules of M of dimension vector $(1, \dots, 1)$ for the quiver Q .*

Note that Q can be chosen to be the Beilinson quiver, J is an ideal just defined by the f_i and M is the unique sincere injective cover of one simple module, in particular M is indecomposable. For more details we refer to section 2. The result on

quiver grassmannians recently attracted attention in connection with cluster algebras (see [10]) and in connection with Auslanders theory on morphisms determined by objects. Ringel has already pointed out that the result above has been studied by several authors ([16]), however 'can be traced back to Beilinson' ([17]). The principal aim of this note is to show how, we can use Beilinson, and even better, how we can even improve it. Eventually, we show that all the constructions at the end can be traced back to a tautological construction in algebraic geometry. Any scheme X is its own Hilbert scheme of sheaves of length one: $\text{Hilb}^1(X) = X$.

We note that the second result, for a variety X , was already stated in [9] and proven again with different methods in [14]. It can certainly be traced back to the work in [4, 5]. An affine version was already proven in [8]. However, the first published result in this direction was just an example in [7]. We will give a common frame for all those examples, in fact all are variants of the Hilbert scheme construction in algebraic geometry and a variant of Beilinsons equivalence. Ringel already noticed that we can even work with the Kronecker quiver, thus, two vertices are sufficient for Q . Improving this construction slightly, we can even realize any projective subscheme of the n -dimensional projective space as a quiver grassmannian for the $(n+1)$ -Kronecker quiver. This construction is again explicit. We denote the m th homogenous component of the ideal I generated by the polynomials f_i by I_m . Thus the homogeneous coordinate ring of X is just $\oplus S^m V / I_m$ for some $(n+1)$ -dimensional vector space V . We also denote by d a natural number greater or equal to the maximal degree of the the polynomials f_i .

Theorem 1.2. *Let X be any projective subscheme of the n -dimensional projective space \mathbb{P}^n . There is a module $M = (S^{d-1}V/I_{d-1}, S^dV/I_d)$ over the Kronecker algebra defined by the natural map $S^{d-1}V/I_{d-1} \otimes V \longrightarrow S^dV/I_d$. Then X is isomorphic to the quiver grassmannian of submodules of M of dimension vector $(1, 1)$.*

The principal part of the note consists of a five step construction that we will use to get a realisation of X as such a moduli space. In addition we also add some modifications of these steps allowing to simplify the quiver or the relations. We explain these five steps briefly. First note, that any scheme X is its own Hilbert scheme of sheaves of length one. So any projective scheme is a moduli space of sheaves (in a rather trivial way). In a second step we use Beilinson's equivalence to construct for X an algebra $A = kQ/J$. Roughly, we can take any tilting bundle T on \mathbb{P} , extend it by any other vector bundle T' to $R = T \oplus T'$ and apply $\text{Hom}(R, -)$ to the universal family of the Hilbert scheme X . In the particular case when T is the direct sum of the line bundles $\mathcal{O}(i)$, for $i = 0, \dots, n$, we can just extend it by the line bundles $\mathcal{O}(i)$ for $i = n+1, \dots, d$. In this way, we get a family of modules over the Beilinson algebra for T and a family of modules over the 'enlarged' Beilinson algebra for R . If X is given by polynomials as above, all modules of the family also satisfy the equations f_i , however now in the Beilinson algebra. Thus we define J to be the ideal generated by the f_i in the enlarged Beilinson algebra. Note that we have several choices for such realization, depending on where the relation starts. However, one can check directly, that family of all modules of dimension vector $(1, \dots, 1)$ over A , the enlarged Beilinson algebra with relations J , coincides with X , independent of the this realization. Consequently, the moduli space of all modules of dimension vector $(1, \dots, 1)$ over A is X as a scheme. In a final step, we realize X as a quiver grassmannian by using an injective hull in A .

We already mentioned that this construction is more general in the way that we can take any tilting bundle T and any vector bundle T' , however the direct computation seems to be more sophisticated. So we use line bundles just for simplicity.

Even stronger, in general we do not need T to be a tilting bundle. For example, the construction also would work if we only take $\mathcal{O} \oplus \mathcal{O}(d)$ where d is at least the maximal degree of the f_i . Then we get a realization similar to Reineke's construction, that is in fact a variant of the realization for the Kronecker quiver.

The paper is organized as follows. In the second section we present the five steps of our construction together with a final note on framed moduli spaces. In the third section we reprove the already known results using the construction in section 2. We conclude in the last section with some open problems and a proof of the second theorem.

2. HILBERT SCHEMES, BEILINSON'S EQUIVALENCE AND SERRE'S CONSTRUCTION

We construct, using some elementary results from algebraic geometry, for any algebraic variety a moduli space of quiver representations, a quiver grassmannian and also further examples in five steps. We also note, that this even holds for any scheme that is quasi-projective. So we obtain the authors example from 1996 [7], Huisgen-Zimmermann's examples in her work on uniserial modules (see for example [8], and her work with Bongartz [4, 5] on Grassmannians, we apologize for being not complete), a variant of Reineke's result for quiver-grassmannians from 2012 [14] and last not least Michel Van den Bergh's example, that appeared in a blog of Lieven le Bruyn. In fact, all the results at the end of this note can be proven using the following constructions.

2.1. Hilbert schemes. ([12])

We take an algebraic variety X and consider sheaves of length one on X . There is a bijection between those sheaves and points of X . In a more sophisticated way we can say $X = \text{Hilb}^1(X)$ the Hilbert scheme of length one sheaves on X . Or, we can consider any line bundle \mathcal{L} on X as a fine moduli space of skyscraper sheaves by taking the push forward to the diagonal in $X \times X$. Each fiber of some point x for the first projection is just the skyscraper sheaf in x .

2.2. Beilinson's tilting bundle. ([2])

In the second step we transform the construction above to the representation theoretic side using tilting. To keep the construction easy, we consider the Beilinson tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1) \oplus \dots \oplus \mathcal{O}(n)$ on the n -dimensional projective space \mathbb{P}^n . We denote by A the Beilinson algebra, it is the opposite of $\text{End}(T)$. Take k_x to be a skyscraper sheaf and let us compute $\text{Hom}(T, k_x)$. Note that $\dim \text{Hom}(\mathcal{O}(i), k_x) = k$, thus we get thin sincere representations of A , that is each simple occurs with multiplicity one. Now we can see by direct computations that the moduli space of thin sincere representations of A is just the projective space \mathbb{P}^n .

This example can be easily generalized to any tilting bundle T , however, using sheaves (that are not vector bundles) we do not get a flat family (the dimension above jumps at certain points).

2.3. Relations of length at most n . ([20])

Now we consider Y , any subvariety defined by equations f_1, \dots, f_r in a projective n -space \mathbb{P}^n . Assume first $\deg f_i \leq n$ for all i . Note that the quiver of the Beilinson algebra above has $n + 1$ arrows from i to $i + 1$, we denote by x_0^i, \dots, x_n^i , and paths the monomials $x_{i(1)}^a x_{i(2)}^{a+1} \dots x_{i(d-1)}^{d-1} x_{i(d)}^d$ of degree $d - a + 1$ at most n . We define an algebra B as the quotient of A by the ideal $J = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r)$, where \bar{f} is any linear combination of path representing f in A . Note that any representative works, since the arrows in A commute, whenever this makes sense: $x_i^a x_j^{a+1} = x_j^a x_i^{a+1}$.

This solves the problem we address to the next step already, since $\mathbb{P}^n \subset \mathbb{P}^m$ for any $n < m$. Moreover, we could also use the fact, that any projective algebraic variety is already defined by quadratic relations. However, using Serre's theorem we can handle also relations of any degree in \mathbb{P}^n .

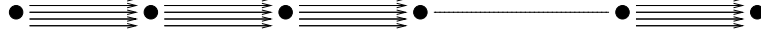


FIGURE 1. Beilinson quiver

2.4. Relations of arbitrary length.

To obtain Y , as in the previous step, where $\deg f_i \leq d$ for any $d > n$, we consider the sequence of line bundles $\mathcal{O}, \dots, \mathcal{O}(d)$. The direct sum of these line bundles is no longer a tilting bundle, however the same computation as above shows that the moduli space of all thin sincere representations of A is still a projective n -space. Now the representatives \bar{f}_i of the polynomials f_i live in $A = \text{End}(\oplus_{i=0}^d \mathcal{O}(i))$ and the moduli space of thin sincere representations of $B = A/(\bar{f}_1, \dots, \bar{f}_r)$ is Y (even scheme theoretic). The reader familiar with Serre's construction will notice that this step is just inspired by this construction ([20]).

2.5. Quiver Grassmannians.

In a final step, we use the embedding of the thin sincere representations of B in its minimal injective hull M . Note that I is the indecomposable injective module with the unique simple socle, that is the socle of any thin sincere representation of B . In case $B = A$ (all f_i are zero) we obtain the projective space as quiver grassmannian of thin sincere subrepresentations of I . In a similar way, also Y coincides with the quiver grassmannian of thin sincere subrepresentations of the large indecomposable injective B -module M . This final step goes back to Schofield [18] and was mentioned later also by Van den Bergh and Ringel ([17]).

2.6. Framed moduli spaces.

We note that quiver grassmannians can be obtained also directly from a corresponding construction in algebraic geometry. Any skyscraper sheaf is the quotient of a line bundle $\mathcal{L} \rightarrow k_x$. If we consider the moduli space of all those quotients of \mathcal{L} with fixed Hilbert series of the quotient sheaf, we get the (framed) Hilbert scheme, that coincides with the original one. If we apply Beilinson's tilting again, we get for $\mathcal{L} = \mathcal{O}(d)$ a projective A -module. Thus \mathbb{P}^n is the quotient grassmannian for the large (that is sincere) indecomposable projective A -module. The same construction works with Y instead of \mathbb{P}^n .

2.7. Reduction of the quiver.

Note that any vector bundle $T = \oplus_{i \in L} \mathcal{O}(i)$ for any L with at least two elements on \mathbb{P}^n defines a morphism from \mathbb{P}^n to the moduli space of modules of dimension vector $(1, 1, \dots, 1, 1)$ over $A = \text{End}(T)$ (or even over the path algebra of A) and also to the corresponding quiver grassmannian for the injective hull M of an indecomposable of dimension vector $(1, 1, \dots, 1, 1)$. Thus, for the projective n -space even two line bundles are sufficient. In the last section we modify this construction slightly and consider L consisting of three, respectively even two, elements so that we still get an isomorphism for any subscheme X in \mathbb{P}^n .

2.8. Proof. We prove the Theorem 1.1 using the five steps above. In explicit terms the module M is just defined by vector spaces $M_m = S^m V / I_m$ the m th graded part of the homogeneous coordinate ring. This becomes a module over the Beilinson quiver using the natural map $V \otimes S^m V \rightarrow S^{m+1} V$ as a multiplication map as follows. Take a basis v_0, \dots, v_n of V and define the linear map of the i th arrow

just by tensoring with v_i : $S^m V \longrightarrow S^{m+1} V$. The commutative relations force that the moduli space (or the corresponding quiver grassmannian in M) of thin sincere modules is just a subscheme in \mathbb{P}^n defined by some of the polynomials f_i . If we consider sufficiently many degrees m , then any f_i is realized in some homogeneous part I_m of the ideal I . For example the two degrees $m = 0, d$ are sufficient to see any f_i . Thus, if we take the Beilinson quiver with vertices $0, 1, 2, \dots, d-1, d$ we can certainly realize the variety X .

Conversely, we may ask how many degrees we need to realize X in the module M . A similar consideration as above shows that the three degrees $m = 0, e, d$ are sufficient, provided d is at least the maximum of all degrees of the polynomials f_i , and e can be any natural number with $0 < e < d$. In particular, we can take $e = 1$ or $e = d - 1$. This leads to the proof of the second theorem proven in the last section. The lemma below reduces than even to the Kronecker quiver.

3. OVERVIEW ON RESULTS AND SOME CONSEQUENCES

3.1. Some variations. Now we can use the construction above to get many variations, we can not list all, however we should collect some. First we construct affine examples. One way is to take open subvarieties, however we would like to characterize open subsets module theoretic. Huisgen-Zimmermann started to consider uniserial modules in [8]. To obtain affine varieties as moduli spaces of uniserial modules, we consider a variant of the Beilinson quiver, we replace the first arrow x_0 just by a path $y_0 z_0$ of length two. Then a thin sincere module is uniserial precisely when its map $y_0 z_0$ is not zero. Thus it is the open subvariety (subscheme) defined by $x_0 = 1$, that is an affine chart.

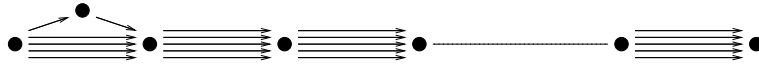


FIGURE 2. modified Beilinson quiver

3.2. Consequences.

In our opinion there are two kind of consequences. First one might think that we can now obtain results in algebraic geometry using representation theory. This seems to be impossible, as far we consider any algebraic variety. However, restricting to some subclasses this might be fruitful, we mention some open problems at the end of this note. Moreover, for our construction, using the Beilinson algebra, the relations are directly given by the defining polynomials. Thus we do not get any deeper insight by considering an algebraic variety as a moduli space of quiver representations.

The second consequence concerns the realisation of a variety as a particular moduli space, that is more restrictive. This is often very useful and is already used quite often. The main open problem here seems to be to construct all moduli spaces of quiver representations for a particular quiver. In general, for all dimension vectors, this is even open for the 3-arrow Kronecker quiver.

3.3. Results. We use the construction in the previous section to prove some of the already known results just by applying the five steps. We start with any projective algebraic variety and proceed with affine ones. As we already explained, we consider X as the scheme of length one sheaves on itself and apply the Beilinson tilting bundle.

Theorem 3.1. [7] *Any projective algebraic scheme of finite type is a fine moduli space of modules over some finite dimensional algebra (a bounded path algebra). Moreover, we can obtain it already for the thin sincere representations, that is the Jordan-Hölder series contains each simple module just once in its composition series up to isomorphism.*

Taking open parts we recover the result of Huisgen-Zimmermann, that was obtained using uniserial modules (Theorem G in [8]). Note, the result was stated in [8] in a different language, the notion of a moduli space was adapted by her only later.

Theorem 3.2. [8] *Any affine algebraic variety is a fine moduli space of uniserial modules over some finite dimensional algebra (a bounded path algebra).*

Then Grassmannians also appeared in Huisgen-Zimmermann's work, however the idea was already introduced by Schofield [18] and then intensively used by Nakajima [13]. However, a similar result could be read off from the work of Bongartz and Huisgen-Zimmermann and was later explicitly stated in [9]. Here again we can use thin sincere submodules of a module M or just uniserial modules.

Theorem 3.3. [9] *Any projective algebraic variety is a quiver grassmannian.*

3.4. Kronecker quiver. Using the Beilinson construction with a rather small vector bundle we can reduce the quiver even to the Kronecker quiver. This is almost the same construction as in Reineke's work and based on the following geometric construction. Consider $\mathbb{P}^n = \mathbb{P}(V)$ embedded into $\mathbb{P}(S^m V)$ with the m -uple embedding. Assume X is a subscheme in $\mathbb{P}(V)$ and consider its image in $\mathbb{P}(S^m V)$. If m is larger than the maximal degree of the polynomials f_i defining X , the equations of X in $\mathbb{P}(S^m V)$ are just linear and the defining equations of the embedding $\mathbb{P}(V) \rightarrow \mathbb{P}(S^m V)$ (that are quadratic). Just modifying the Beilinson construction we can use the bundle $\mathcal{O} \oplus \mathcal{O}(e) \oplus \mathcal{O}(d)$. This reduces the construction to a quiver with three vertices. For $e = d/2$ and d sufficiently large, this corresponds to realizing X using quadratic equations. The corresponding module M considered as a representation of a three vertex quiver (M_1, M_2, M_3) has a simple socle $M_3 = k = S^0 V$ with $M_2 = S^e V$ and $M_1 = S^d V$. Now we use Ringel's idea to reduce to the Kronecker quiver $S^{d-e} V$.

Lemma 3.4. *With notation above and any $d > e > 0$ we have an isomorphism of quiver grassmannians as follows. The quiver grassmannian of submodules of $M = (M_1, M_2, M_3)$ of dimension vector $(1, 1, 1)$ is isomorphic to the quiver grassmannian of submodules of (M_1, M_2) of dimension vector $(1, 1)$.*

PROOF. Note that the restriction of M to (M_1, M_2) defines a morphism of quiver grassmannians. Since M_3 is just one-dimensional, any submodule (M_1, M_2) over the Kronecker algebra of dimension vector $(1, 1)$ extends uniquely to a submodule of M of dimension vector $(1, 1, 1)$. Thus, this morphism is a bijection. In the particular case of X being the projective space, this morphism is an isomorphism. Going back to M we just restrict this isomorphism to the subscheme defined by the polynomials f_i , consequently, both quiver grassmannians are also isomorphic. \square

Taking d at least the degree of the defining equations f_i and $d = e + 1$ we realize X as a quiver grassmannian over the Kronecker algebra with $(n + 1)$ vertices. This proves Theorem 1.2. Note that Reineke realized the linear subspace using an additional arrow, however, this is not necessary.

Obviously, we can not reduce to just one vertex, thus two vertices is the minimum we can achieve. However, it is not clear whether we can still reduce the number of

arrows. Such a reduction would be more complicated and certainly independent of Beilinson's result.

3.5. Affine versus projective. At the end we discuss the problem how to obtain projective examples from affine ones and vice versa. As we have mentioned above, one can take a projective variety, that is a moduli space, and obtain an affine cover as moduli spaces of uniserials by modifying the Beilinson quiver slightly.

The converse, to obtain complete examples by glueing, is an open problem. In particular, let X be a complete variety that is not projective (see Hartshorne for an example [6], Ex 3.4.1 in appendix B) then to our knowledge there is no way so far, to get X as a moduli space of representations. Moreover, it is clear that X can not be a quiver grassmannian, since the latter one is projective by definition. One might think that also moduli spaces are always projective, however, we should mention that moduli spaces as constructed in King's paper [11] are, but there might be other constructions as well.

3.6. Further open problems. Since already Kronecker quivers are very complicated with respect to the geometry of quiver grassmannians it would be natural to restrict to particular classes of modules or quivers. As far we know, the problem to describe all quiver grassmannians is open for Dynkin quivers and also tame quivers. It also would be desirable to understand quiver grassmannians for the 3-arrow Kronecker quiver. Moreover, inspired by cluster algebras, the main open problem seems to be to understand quiver grassmannians for exceptional modules over path algebras.

If we use the explicit construction of the module M with $M_m = S^m V / I_m$ one can see, that everything is even defined over any base field. For polynomials over the integers everything is defined even over \mathbb{Z} . Thus the construction also works in the same fashion over an commutative ring.

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